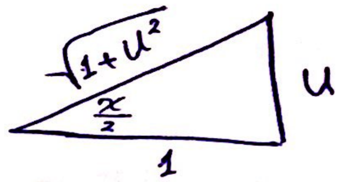
3. $J = \int \frac{dx}{1 + \sin x - \cos x}$



Ans: Put $u = \tan\frac{x}{2} \Rightarrow du = \frac{1}{2}\sec^2\frac{x}{2} \cdot dx$

$\sec^2\frac{x}{2} = \frac{1}{\cos^2\frac{x}{2}} = 1 + u^2$

$\therefore \boxed{dx = \frac{2\,du}{1+u^2}}$,

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$\sin x = 2\sin\frac{x}{2} \cdot \cos\frac{x}{2} = 2 \cdot \frac{u}{\sqrt{1+u^2}} \cdot \frac{1}{\sqrt{1+u^2}}$, Then

$\boxed{\sin x = \frac{2u}{1+u^2}}$,

$\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{1+u^2} - 1 = \frac{2-1-u^2}{1+u^2}$, Then

$\boxed{\cos x = \frac{1-u^2}{1+u^2}}$, Then

$$J = \int \frac{\frac{2\,du}{1+u^2}}{1 + \frac{2u}{1+u^2} - \frac{1-u^2}{1+u^2}} = 2\int \frac{du}{1+u^2+2u-1+u^2} = 2\int \frac{du}{2u^2+2u}$$

$= \int \frac{du}{u(u+1)}$ using Partial fraction, Then

$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$

$A(u+1) + B \cdot u = 1$

$A = 1$, $B = -1$, Then

$= \int \frac{du}{u} - \int \frac{du}{u+1}$

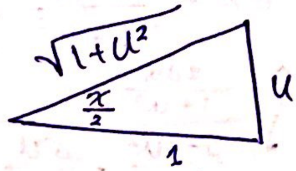
$= \ln|u| - \ln|u+1| + C = \ln\left|\frac{u}{u+1}\right| + C$

$= \boxed{\ln\left|\frac{\tan\frac{x}{2}}{\tan\frac{x}{2}+1}\right| + C}$

(4) $I = \int \frac{dx}{\sin x - \tan x}$

Ans:

$$I = \int \frac{dx}{\sin x - \frac{\sin x}{\cos x}} = \int \frac{\cos x}{\sin x \cdot \cos x - \sin x}\, dx$$



Put $u = \tan\frac{x}{2} \Rightarrow du = \frac{1}{2}\sec^2 \frac{x}{2}\, dx$

$\boxed{dx = \frac{2\,du}{1+u^2}}$, $\sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}$

$\boxed{\sin x = \frac{2u}{1+u^2}}$, $\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{1+u^2} - 1$

$\boxed{\cos x = \frac{1-u^2}{1+u^2}}$, Then

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$$I = \int \frac{\frac{1-u^2}{1+u^2}\cdot\frac{2\,du}{1+u^2}}{\frac{2u(1-u^2)}{(1+u^2)^2} - \frac{2u}{1+u^2}} = \int \frac{2(1-u^2)}{2u(1-u^2) - 2u(1+u^2)}\, du$$

$$= \int \frac{1-u^2}{u-u^3-u-u^3}\, du = \int \frac{1-u^2}{-2u^3}\, du = \frac{-1}{2}\left[\int \frac{du}{u^3} - \int \frac{du}{u}\right]$$

$$= \frac{-1}{2}\left[\frac{u^{-2}}{-2} - \ln|u|\right] + C = \frac{1}{4u^2} + \frac{1}{2}\ln|u| + C$$

$$= \boxed{\frac{1}{4\tan^2\frac{x}{2}} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C}$$

(5) $I = \int \frac{x^3}{\sqrt[3]{9-x^2}}\, dx$

Ans: Put $u = \sqrt[3]{9-x^2} \Rightarrow u^3 = 9 - x^2 \Rightarrow x^2 = 9 - u^3$, Then

$2x\, dx = -3u^2\, du$, Then

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$$I = \int \frac{x^2 \cdot x \cdot dx}{\sqrt[3]{9-x^2}} = \frac{-3}{2}\int \frac{(9-u^3)\, u^{\cancel{2}}\, du}{\cancel{u}}$$

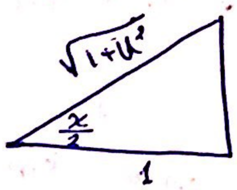
$$= \frac{-3}{2}\int (9u - u^4)\, du = \frac{-3}{2}\left(\frac{9}{2}u^2 - \frac{1}{5}u^5\right) + C$$

$$= \boxed{\frac{3}{10}(9-x^2)^{\frac{5}{3}} - \frac{27}{4}(9-x^2)^{\frac{2}{3}} + C}$$

(42) $I = \int \frac{dx}{5\cos x - 3\sin x + 1}$

Ans: Put $u = \tan\frac{x}{2} \Rightarrow du = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$\therefore dx = \frac{2\,du}{1+u^2}$, Then



$$I = \int \frac{\frac{2du}{1+u^2}}{\frac{5-5u^2}{1+u^2} - \frac{6u}{1+u^2} + 1} * \frac{1+u^2}{1+u^2}$$

$* \sec^2\frac{x}{2} = 1 + u^2$

$* \cos x = 2\cos^2\frac{x}{2} - 1$

$= \frac{1-u^2}{1+u^2}$

$* \sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}$

$= \frac{2u}{1+u^2}$

$$= \int \frac{2}{5-5u^2-6u+1+u^2}\,du$$

$$= -2\int \frac{du}{4u^2+6u-6}$$

$* 4u^2+6u-6 = 4\left(u^2+\frac{3}{2}u-\frac{3}{2}\right)$

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$= 4\left[\left(u+\frac{3}{4}\right)^2 - \frac{9}{16} - \frac{3}{2}\right]$

$= 4\left[\left(u+\frac{3}{4}\right)^2 - \frac{33}{16}\right]$

$$= \frac{1}{2}\int \frac{du}{\frac{33}{16} - \left(u+\frac{3}{4}\right)^2} =$$

$$= \frac{1}{2}\int \frac{du}{a^2 - z^2} = \frac{1}{2}\cdot\frac{1}{a}\tanh^{-1}\left(\frac{z}{a}\right) + C$$

$$= \frac{1}{2}\cdot\frac{4}{\sqrt{33}}\tanh^{-1}\left(\frac{4\left(u+\frac{3}{4}\right)}{\sqrt{33}}\right) + C.$$

$$= \boxed{\frac{2}{\sqrt{33}}\tanh^{-1}\left(\frac{4\tan\left(\frac{x}{2}\right)+3}{\sqrt{33}}\right) + C}$$

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(6) $J = \int \frac{dx}{\sqrt{1-e^x}}$

Ans: Put $u = \sqrt{1-e^x} \Rightarrow u^2 = 1 - e^x \Rightarrow e^x = 1 - u^2$

$e^x \, dx = -2u \, du \Rightarrow dx = \frac{-2u \, du}{e^x} = \frac{-2u \, du}{1-u^2}$, Then

$$J = \int \frac{\frac{-2\cancel{u} \, du}{1-u^2}}{\cancel{u}} = -2 \int \frac{du}{1-u^2} = -2 \tanh^{-1}(u) + C$$

$$= \boxed{-2 \tanh^{-1}\left[\sqrt{1-e^x}\right] + C}$$

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(7) $I = \int \frac{dx}{x^2 \cdot \sqrt{x^2+2x}}$

Ans: Put $u = \frac{1}{x} \Rightarrow x = \frac{1}{u} \Rightarrow dx = \frac{-du}{u^2} \Rightarrow x^2 = \frac{1}{u^2}$, Then

$$I = \int \frac{\frac{-du}{u^2}}{\frac{1}{u^2} \cdot \sqrt{\frac{1}{u^2} + \frac{2}{u}}} = -\int \frac{u \, du}{\sqrt{1+2u}}$$

let $z = \sqrt{1+2u} \Rightarrow z^2 = 1 + 2u$

$\cancel{2} z \, dz = \cancel{2} \, du$

$$I = -\int \frac{(z^2-1) \cdot \cancel{z} \, dz}{2\cancel{z}} = \frac{-1}{2}\left[\frac{1}{3} z^3 - z\right] + C$$

$$= \frac{-1}{6}(1+2u)^{\frac{3}{2}} + \frac{1}{2}(1+2u)^{\frac{1}{2}} + C$$

$$= \frac{1}{6}(1+2u)^{\frac{1}{2}}\left[3 - (1+2u\right] + C$$

$$= \frac{1}{3} \cdot \sqrt{1+2u}\,(1-u) + C$$

$$= \frac{1}{3} \cdot \sqrt{1+\frac{2}{x}} \cdot \left(1 - \frac{1}{x}\right) + C = \boxed{\frac{x-1}{3x^2}\sqrt{x^2+2x} + C}$$